

# ERLANGEN PROGRAM AT LARGE— $2\frac{1}{2}$ : INDUCED REPRESENTATIONS AND HYPERCOMPLEX NUMBERS

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**ABSTRACT.** We review the construction of induced representations of the group  $G = \mathrm{SL}_2(\mathbb{R})$ . Firstly we note that  $G$ -action on the homogeneous space  $G/H$ , where  $H$  is any one-dimensional subgroup of  $\mathrm{SL}_2(\mathbb{R})$ , is a linear-fractional transformation on hypercomplex numbers. Thus we investigate various hypercomplex characters of subgroups  $H$ . Finally we give examples of induced representations of  $\mathrm{SL}_2(\mathbb{R})$  on spaces of hypercomplex valued functions, which are unitary in some sense.

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## 1. THE GROUP $\mathrm{SL}_2(\mathbb{R})$ AND ITS SUBGROUPS

Let  $\mathrm{SL}_2(\mathbb{R})$  be **the group of  $2 \times 2$  matrices** with real entries and of determinant one [26]. This is the smallest semisimple Lie group. Any matrix in  $\mathrm{SL}_2(\mathbb{R})$  admits a (unique) decomposition of the form [10, Exer. I.14]:

$$(1) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix},$$

2000 *Mathematics Subject Classification.* Primary 22D30; Secondary 08A99, 13A50, 15A04, 20H05, 51M10.

*Key words and phrases.* induced representation, unitary representations,  $\mathrm{SL}_2(\mathbb{R})$ , semisimple Lie group, complex numbers, dual numbers, double numbers, Möbius transformations, split-complex numbers, parabolic numbers, hyperbolic numbers.

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for some values  $\alpha \in (0, \infty)$ ,  $\nu \in (-\infty, \infty)$  and  $\phi \in (-\pi, \pi]$ . The formula (1) rewritten in a way  $SL_2(\mathbb{R}) = ANK$  is known as Iwasawa decomposition [26, § III.1] and can be generalised to any semisimple Lie group.

Each out of the three types of matrices in the right-hand side of (1) forms a one-parameter subgroup A, N and K. They are obtained by the exponentiation of the respective zero-trace matrices:

$$(2) \quad A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = \exp \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}, t \in \mathbb{R} \right\},$$

$$(3) \quad N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}, t \in \mathbb{R} \right\},$$

$$(4) \quad K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}, t \in (-\pi, \pi] \right\}.$$

The following simple result have an instructive proof.

**Proposition 1.** *Any continuous one-parameter subgroup of  $SL_2(\mathbb{R})$  is conjugate to one of subgroups A, N or K.*

*Proof.* Any one-parameter subgroup is obtained through the exponentiation

$$(5) \quad e^{tX} = \sum_{n=0}^{\infty} \frac{t^n}{n!} X^n$$

of an element X of the Lie algebra  $\mathfrak{sl}_2$  of  $SL_2(\mathbb{R})$ . Such X is a  $2 \times 2$  matrix with the zero trace. The behaviour of the Taylor expansion (5) depends from properties of powers  $X^n$ . This can be classified by a straightforward calculation:

**Lemma 2.** *The square  $X^2$  of a traceless matrix  $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$  is the identity matrix times  $a^2 + bc = -\det X$ . The factor can be negative, zero or positive, which corresponds to the three different types of the Taylor expansion (5) of  $e^{tX}$ .*

It is a simple exercise in the Gauss elimination to see that through the matrix similarity we can obtain from X a generator

- of the subgroup K if  $(-\det X) < 0$ ;
- of the subgroup N if  $(-\det X) = 0$ ;
- of the subgroup A if  $(-\det X) > 0$ .

The determinant is invariant under the similarity, thus these cases are distinct.  $\square$

**Example 3.** The following two subgroups are conjugated to A and N respectively:

$$(6) \quad A' = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} = \exp \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, t \in \mathbb{R} \right\},$$

$$(7) \quad N' = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \exp \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}, t \in \mathbb{R} \right\}.$$

## 2. ACTION OF $SL_2(\mathbb{R})$ AS A SOURCE OF HYPERCOMPLEX NUMBERS

Let H be a subgroup of a group G. Let  $\Omega = G/H$  be the corresponding homogeneous space and  $s : \Omega \rightarrow G$  be a smooth section [12, § 13.2], which is a left inverse to the natural projection  $p : G \rightarrow \Omega$ . The choice of s is inessential in the sense that by a smooth map  $\Omega \rightarrow \Omega$  we can always reduce one to another.

Any  $g \in G$  has a unique decomposition of the form  $g = s(\omega)h$ , where  $\omega = p(g) \in \Omega$  and  $h \in H$ . Note that  $\Omega$  is a left homogeneous space with the G-action defined in terms of p and s as follows:

$$(8) \quad g : \omega \mapsto g \cdot \omega = p(g * s(\omega)),$$

where  $*$  is the multiplication on  $G$ . This is also illustrated by the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{g^*} & G \\ s \downarrow & & \downarrow p \\ \Omega & \xrightarrow{g \cdot} & \Omega \end{array}$$

For  $G = \mathrm{SL}_2(\mathbb{R})$ , as well as for other semisimple groups, it is common to consider only the case of  $H$  being the maximal compact subgroup  $K$ . However in this paper we admit  $H$  to be any one-dimensional subgroup. Due to the previous Proposition it is sufficient to take  $H = K$ ,  $N'$  or  $A'$ . Then  $\Omega$  is a two-dimensional manifold and for any choice of  $H$  we define [13, Ex. 3.7(a)]:

$$(9) \quad s : (u, v) \mapsto \frac{1}{\sqrt{v}} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2, v > 0.$$

A direct (or computer algebra [21]) calculation show that:

**Proposition 4.** The  $\mathrm{SL}_2(\mathbb{R})$  action (8) associated to the map  $s$  (9) is:

$$(10) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u, v) \mapsto \left( \frac{(au + b)(cu + d) - \sigma c v^2}{(cu + d)^2 - \sigma (cv)^2}, \frac{v}{(cu + d)^2 - \sigma (cv)^2} \right),$$

where  $\sigma = -1, 0$  and  $1$  for the subgroups  $K, N'$  and  $A'$  respectively.

The expression in (10) does not look very appealing, however an introduction of hypercomplex numbers makes it more attractive:

**Proposition 5.** Let an imaginary unit  $\iota$  is such that  $\iota^2 = \sigma$ , then the  $\mathrm{SL}_2(\mathbb{R})$  action (10) becomes:

$$(11) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \frac{aw + b}{cw + d}, \quad \text{where } w = u + \iota v,$$

for all three cases parametrised by  $\sigma$  as in Prop. 4.

The imaginary unit  $i^2 = -1$  is the very well-known case of the complex numbers. The two-dimensional algebra of numbers  $x + \epsilon y$  with the imaginary unit  $\epsilon^2 = 1$  is known as split-complex, hyperbolic or *double numbers* [3, 11, 31], they are one of simplest cases of *hypernumbers*. The parabolic hypercomplex numbers also called as *dual numbers* are of the form  $x + \epsilon y$  with the imaginary unit<sup>1</sup> such that  $\epsilon^2 = 0$  [4, 9, 33]. In cases when we need to consider several imaginary units simultaneously we use  $\iota$  to denote any of  $i, \epsilon$  or  $\epsilon$ .

*Remark 6.* The parabolic imaginary unit  $\epsilon$  is a close relative to the infinitesimal number  $\epsilon$  from non-standard analysis [5, 32]. The former has the property that its square is *exactly* zero, meanwhile the square of the latter is *almost* zero at its own scale. This similarity is exploit in [4] to produce non-standard proofs of the main calculus theorems. A similar property allows to obtain classical mechanics from the representations of the Heisenberg group [22].

Notably the action (11) is a group homomorphism of the group  $\mathrm{SL}_2(\mathbb{R})$  into transformations of the “upper half-plane” on hypercomplex numbers. Although dual and double numbers are algebraically trivial, the respective geometries in the spirit of *Erlangen programme* are refreshingly inspiring [15] and provide useful insights even in the elliptic case [17]. In order to treat divisors of zero, we need to consider Möbius transformations (11) of conformally completed plane [9, 19].

<sup>1</sup>We use different scripts of the epsilon:  $\epsilon$  denote hyperbolic imaginary unit and  $\epsilon$ —parabolic one.

Physical applications of hypercomplex numbers are scattered through classical mechanics [33], relativity [3,31], cosmology [7,9] and quantum mechanics [11,22].

Now we wish to linearise the action (8) through the **induced representations** [12, § 13.2; 13, § 3.1]. We define a map  $r : G \rightarrow H$  associated to  $p$  and  $s$  from the identities:

$$(12) \quad r(g) = (s(\omega))^{-1}g, \quad \text{where } \omega = p(g) \in \Omega.$$

Let  $\chi$  be an irreducible representation of  $H$  in a vector space  $V$ , then it induces a representation of  $G$  in the sense of Mackey [12, § 13.2]. This representation has the realisation  $\rho_\chi$  in the space of  $V$ -valued functions by the formula [12, § 13.2.(7)–(9)]:

$$(13) \quad [\rho_\chi(g)f](\omega) = \chi(r(g^{-1} * s(\omega)))f(g^{-1} \cdot \omega),$$

where  $g \in G$ ,  $\omega \in \Omega$ ,  $h \in H$  and  $r : G \rightarrow H$ ,  $s : \Omega \rightarrow G$  are maps defined above;  $*$  denotes multiplication on  $G$  and  $\cdot$  denotes the action (8) of  $G$  on  $\Omega$ .

In our consideration  $H$  is always one-dimensional, its irreducible representation is always supposed to be a complex valued character. However hypercomplex number naturally appear in the  $SL_2(\mathbb{R})$  action (11), why shall we admit only  $i^2 = -1$  to deliver a character then?

### 3. HYPERCOMPLEX CHARACTERS—AN ALGEBRAIC APPROACH

As we already mentioned the typical discussion of induced representations of  $SL_2(\mathbb{R})$  is centred around the case  $H = K$  and a complex valued character of  $K$ . A linear transformation defined by a matrix (4) in  $K$  is a rotation of  $\mathbb{R}^2$  by the angle  $t$ . After identification  $\mathbb{R}^2 = \mathbb{C}$  this action is given by the multiplication  $e^{it}$ , with  $i^2 = -1$ . The rotation preserve the (elliptic) metric given by:

$$(14) \quad x^2 + y^2 = (x + iy)(x - iy).$$

Therefore the orbits of rotations are circles, any line passing the origin (a “spoke”) is rotated by the angle  $t$ , see Fig. 1(E).

Introduction of hypercomplex numbers produces the most straightforward adaptation of this result.

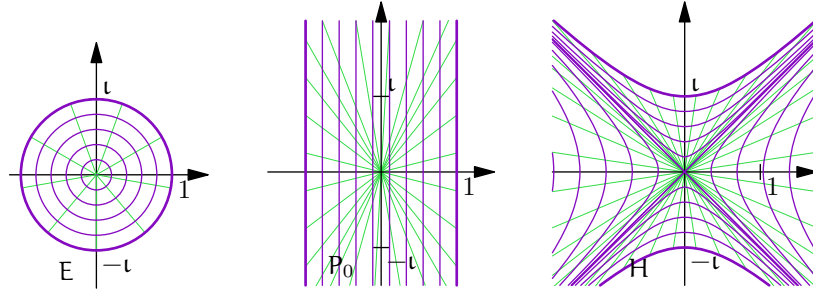


FIGURE 1. Rotations of algebraic wheels, i.e. the multiplication by  $e^{it}$ : elliptic (E), trivial parabolic ( $P_0$ ) and hyperbolic (H). All blue orbits are defined by the identity  $x^2 - t^2 y^2 = r^2$ . Green “spokes” (straight lines from the origin to a point on the orbit) are “rotated” from the real axis.

**Proposition 7.** *The following table show correspondences between three types of algebraic characters:*

<i>Elliptic</i>	<i>Parabolic</i>	<i>Hyperbolic</i>
$i^2 = -1$	$\varepsilon^2 = 0$	$\epsilon^2 = 1$
$w = x + iy$	$w = x + \varepsilon y$	$w = x + \epsilon y$
$\bar{w} = x - iy$	$\bar{w} = x - \varepsilon y$	$\bar{w} = x - \epsilon y$
$e^{it} = \cos t + i \sin t$	$e^{\varepsilon t} = 1 + \varepsilon t$	$e^{\epsilon t} = \cosh t + \epsilon \sinh t$
$ w _e^2 = w\bar{w} = x^2 + y^2$	$ w _p^2 = w\bar{w} = x^2$	$ w _h^2 = w\bar{w} = x^2 - y^2$
$\arg w = \tan^{-1} \frac{y}{x}$	$\arg w = \frac{y}{x}$	$\arg w = \tanh^{-1} \frac{y}{x}$
<i>unit circle</i> $ w _e^2 = 1$	<i>“unit” strip</i> $x = \pm 1$	<i>unit hyperbola</i> $ w _h^2 = 1$

Geometrical action of multiplication by  $e^{it}$  is drawn on Fig. 1 for all three cases.

Explicitly parabolic rotations associated with  $e^{\varepsilon t}$  acts on dual numbers as follows:

$$(15) \quad e^{\varepsilon x} : a + \varepsilon b \mapsto a + \varepsilon(ax + b).$$

This links the parabolic case with the Galilean group [33] of symmetries of the classic mechanics, with the absolute time disconnected from space.

The obvious algebraic similarity and the connection to classical kinematic is a wide spread justification for the following viewpoint on the parabolic case, cf. [8, 33]:

- the parabolic trigonometric functions are trivial:

$$(16) \quad \csc t = \pm 1, \quad \sin t = t;$$

- the parabolic distance is independent from  $y$  if  $x \neq 0$ :

$$(17) \quad x^2 = (x + \varepsilon y)(x - \varepsilon y);$$

- the polar decomposition of a dual number is defined by [33, App. C(30')]:

$$(18) \quad u + \varepsilon v = u(1 + \varepsilon \frac{v}{u}), \quad \text{thus} \quad |u + \varepsilon v| = u, \quad \arg(u + \varepsilon v) = \frac{v}{u};$$

- the parabolic wheel looks rectangular, see Fig. 1(P<sub>0</sub>).

Those algebraic analogies are quite explicit and widely accepted as an ultimate source for parabolic trigonometry [8, 27, 33]. However we will see shortly that there exist geometric motivation and connection with parabolic equation of mathematical physics.

#### 4. A PARABOLIC WHEEL—A GEOMETRICAL VIEWPOINT

We make another attempt to describe parabolic rotations. If multiplication (a linear transformation) is not sophisticated enough for this we can advance to the next level of complexity: linear-fractional.

Imaginary units do not need to be seen as abstract quantities. As follows from Lem. 2 the generators of subgroup  $K$ ,  $N$  and  $A$  represent imaginary units of complex, dual and double numbers respectively. Their exponentiation to one-parameter subgroups  $K$ ,  $N'$  and  $A'$  of  $SL_2(\mathbb{R})$  produce matrix forms of the Euler identities from the fifth row of the table in Prop. 7.

Thus we attempt to define characters of subgroups  $K$ ,  $N'$  and  $A'$  in term of geometric action of  $SL_2(\mathbb{R})$  by Möbius transformations. The action (11) is defined on the upper half-plane and to relate it to unitary characters we wish to transfer it

to the unit disk. In the elliptic case this is done by the Cayley transform, its action on the subgroup K is:

$$(19) \quad \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}.$$

The diagonal matrix in the right hand side define the Moebius transformation which reduces to multiplication by  $e^{2it}$ , i.e. the elliptic rotation.

A hyperbolic cousin of the Cayley transform is:

$$(20) \quad \frac{1}{2} \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix} = \begin{pmatrix} e^{\epsilon t} & 0 \\ 0 & e^{-\epsilon t} \end{pmatrix},$$

similarly produces a Moebius transformation which is the multiplication by  $e^{2\epsilon t}$ , which a unitary (Lorentz) transformation of two-dimensional Minkowski space-time.

In the parabolic case we use the similar pattern and define the Cayley transform from the matrix:

$$C_\epsilon = \begin{pmatrix} 1 & -\epsilon \\ -\epsilon & 1 \end{pmatrix}$$

The Cayley transform of matrices (3) from the subgroup N is:

$$(21) \quad \begin{pmatrix} 1 & -\epsilon \\ -\epsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix} = \begin{pmatrix} 1 + \epsilon t & t \\ 0 & 1 - \epsilon t \end{pmatrix} = \begin{pmatrix} e^{\epsilon t} & t \\ 0 & e^{-\epsilon t} \end{pmatrix}.$$

This is not far from the previous identities (19) and (20), however, the off-diagonal (1, 2)-term destroys harmony. Nevertheless we will continue a unitary parabolic rotation to be the Möbius transformation with the matrix (21), which will not be a multiplication by a scalar anymore.

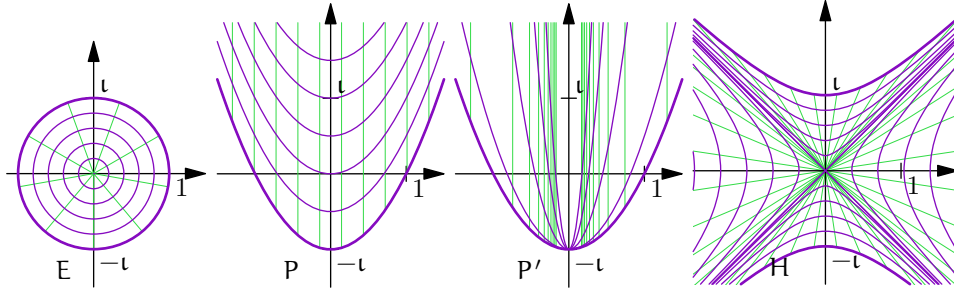


FIGURE 2. Rotation of geometric wheels: elliptic (E), two parabolic (P and P') and hyperbolic (H). Blue orbits are level lines for the respective moduli. Green straight lines join points with the same value of argument and are drawn with the constant "angular step" in each case.

**Example 8.** The parabolic rotations with the upper-triangular matrices from the subgroup N becomes:

$$(22) \quad \begin{pmatrix} e^{\epsilon t} & t \\ 0 & e^{-\epsilon t} \end{pmatrix} : -\epsilon \mapsto t + \epsilon(t^2 - 1).$$

This coincides with the *cyclic rotations* defined in [33, § 8]. A comparison with the Euler formula seemingly confirms that  $\sin p t = t$ , but suggests a new expression for  $\csc p t$ :

$$\csc p t = 1 - t^2, \quad \sin p t = t.$$

Therefore the parabolic Pythagoras' identity would be:

$$(23) \quad \sin^2 t + \csc^2 t = 1,$$

which nicely fits in between the elliptic and hyperbolic versions:

$$\sin^2 t + \cos^2 t = 1, \quad \sinh^2 t - \cosh^2 t = -1.$$

The identity (23) is also less trivial than the version  $\csc^2 t = 1$  from [8] (see also (16), (17)).

**Example 9.** There is the second option to define parabolic rotations for the lower-triangular matrices from the subgroup  $N'$ . The important difference now is: the reference point cannot be  $-\varepsilon$  since it is a fixed point (as well as any point on the vertical axis). Instead we take  $\varepsilon^{-1}$ , which is an ideal element (a point at infinity [33, App. C]) since  $\varepsilon$  is a divisor of zero. The proper compactifications by ideal elements for all three cases were discussed in [19].

We get for the subgroup  $N'$ :

$$(24) \quad \begin{pmatrix} e^{-\varepsilon t} & 0 \\ t & e^{\varepsilon t} \end{pmatrix} : \frac{1}{\varepsilon} \mapsto \frac{1}{t} + \varepsilon \left(1 - \frac{1}{t^2}\right).$$

A comparison with (22) shows that this form is obtained by the change  $t \mapsto t^{-1}$ . The same transformation gives new expressions for parabolic trigonometric functions. The parabolic “unit circle” (or *cycle* [15, 33]) is defined by the equation  $x^2 - y = 1$  in both cases, see Fig. 2(P) and (P'). However other orbits are different and we will give their description in the next Section.

Fig. 2 illustrates Möbius actions of matrices (19), (20) and (21) on the respective “unit disk”, which are images of the upper half-planes under respective Cayley transforms [15, § 8].

## 5. REBUILDING ALGEBRAIC STRUCTURES FROM GEOMETRY

We want induced representations to be linear, to this end the inducing character shall be linear as well. Rotations in elliptic and hyperbolic cases are given by products of complex or double numbers respectively and thus are linear. However non-trivial parabolic rotations (22) and (24) (Fig. 2(P) and (P')) are not linear. Can we find algebraic operations for dual numbers, which will linearise those Möbius transformations?

It is common in mathematics to “revert a theorem into a definition” and we will use this systematically to recover a compatible algebraic structure.

**5.1. Modulus and Argument.** In the elliptic and hyperbolic cases orbits of rotations are points with the constant norm (modulus): either  $x^2 + y^2$  or  $x^2 - y^2$ . In the parabolic case we employ this point of view as well:

**Definition 10.** Orbits of actions (22) and (24) are contour lines for the following functions which we call respective moduli (norms):

$$(25) \quad \text{for } N : |u + \varepsilon v| = u^2 - v, \quad \text{for } N' : |u + \varepsilon v|' = \frac{u^2}{v + 1}.$$

*Remark 11.* (1) The expression  $|(u, v)| = u^2 - v$  represents a parabolic distance from  $(0, \frac{1}{2})$  to  $(u, v)$ , see [15, Lem. 8.4], which is in line with the “parabolic Pythagoras' identity” (23).

(2) Modulus for  $N'$  expresses the parabolic focal length from  $(0, -1)$  to  $(u, v)$  as described in [15, Lem. 8.5].



The only straight lines preserved by both the parabolic rotations  $N$  and  $N'$  are vertical lines, thus we will treat them as “spokes” for parabolic “wheels”. Elliptic spokes in mathematical terms are “points on the complex plane with the same argument”, thus we again use this for the parabolic definition:

**Definition 12.** Parabolic arguments are defined as follows:

$$(26) \quad \text{for } N : \arg(u + \varepsilon v) = u, \quad \text{for } N' : \arg'(u + \varepsilon v) = \frac{1}{u}.$$

Both Definitions 10 and 12 possess natural properties with respect to parabolic rotations:

**Proposition 13.** Let  $w_t$  be a parabolic rotation of  $w$  by an angle  $t$  in (22) or in (24). Then:

$$|w_t|^{(r)} = |w|^{(r)}, \quad \arg^{(r)} w_t = \arg^{(r)} w + t,$$

where primed versions are used for subgroup  $N'$ .

All proofs in this and the following Sections were performed through symbolic calculations on a computer. See Appendices A–B for details.

*Remark 14.* Note that in the commonly accepted approach [33, App. C(30')] parabolic modulus and argument are given by expressions (18), which are, in a sense, opposite to our agreements.

**5.2. Rotation as Multiplication.** We revert again theorems into definitions to assign multiplication. In fact, we consider parabolic rotations as multiplications by unimodular numbers thus we define multiplication through an extension of properties from Proposition 13:

**Definition 15.** The product of vectors  $w_1$  and  $w_2$  is defined by the following two conditions:

- (1)  $\arg^{(r)}(w_1 w_2) = \arg^{(r)} w_1 + \arg^{(r)} w_2;$
- (2)  $|w_1 w_2|^{(r)} = |w_1|^{(r)} \cdot |w_2|^{(r)}.$

We also need a special form of parabolic conjugation, which coincides with sign reversion of the argument.

**Definition 16.** Parabolic conjugation is given by

$$(27) \quad \overline{u + \varepsilon v} = -u + \varepsilon v.$$

Obviously we have the properties:  $|\overline{w}|^{(r)} = |w|^{(r)}$  and  $\arg^{(r)} \overline{w} = -\arg^{(r)} w$ . A combination of Definitions 10, 12 and 15 uniquely determine expressions for products.

**Proposition 17.** The parabolic product of vectors is defined by formulae:

$$(28) \quad \text{for } N : (u, v) * (u', v') = (u + u', (u + u')^2 - (u^2 - v)(u'^2 - v'));$$

$$(29) \quad \text{for } N' : (u, v) * (u', v') = \left( \frac{uu'}{u + u'}, \frac{(v + 1)(v' + 1)}{(u + u')^2} - 1 \right).$$

Although both expressions look unusual they have many familiar properties:

**Proposition 18.** Both products (28) and (29) satisfy the following conditions:

- (1) They are commutative and associative;
- (2) The respective rotations (22) and (24) are given by multiplications with a dual number with the unit norm.
- (3) The product  $w_1 \bar{w}_2$  is invariant under respective rotations (22) and (24).
- (4) For any dual number  $w$  the following identity holds:

$$|w \bar{w}| = |w|^2.$$



In particular, the property (3) will be crucial below for an inner product (35), which makes induced representations unitary.

## 6. INVARIANT LINEAR ALGEBRA

Now we wish to define a linear structure on  $\mathbb{R}^2$  which would be invariant under point multiplication from the previous Subsection (and thus under the parabolic rotations, cf. Prop. 18(2)). Multiplication by a real scalar is straightforward (at least for a positive scalar): it should preserve the argument and scale the norm of a vector. Thus we have formulae for  $\alpha > 0$ :

$$(30) \quad \alpha \cdot (u, v) = (u, \alpha v + u^2(1 - \alpha)) \quad \text{for } N,$$

$$(31) \quad \alpha \cdot (u, v) = \left( u, \frac{v+1}{\alpha} - 1 \right) \quad \text{for } N'.$$

On the other hand the addition of vectors can be done in several different ways. We present two possibilities: one is tropical and another—exotic.

**6.1. Tropical form.** Let us introduce the lexicographic order on  $\mathbb{R}^2$ :

$$(u, v) \prec (u', v') \quad \text{if and only if} \quad \begin{cases} \text{either } u < u'; \\ \text{or } u = u', v < v'. \end{cases}$$

One can define functions  $\min$  and  $\max$  of a pair of points on  $\mathbb{R}^2$  respectively. Then an addition of two vectors can be defined either as their minimum or maximum. A similar definition is used in *tropical mathematics*, also known as Maslov dequantisation or  $\mathbb{R}_{\min}$  and  $\mathbb{R}_{\max}$  algebras, see [28] for a comprehensive survey. It is easy to check that such an addition is distributive with respect to scalar multiplications (30)—(31) and consequently is invariant under parabolic rotations. Although it looks promising to investigate this framework we do not study it further for now.

**6.2. Exotic form.** Addition of vectors for both subgroups  $N$  and  $N'$  can be defined by the common rules, where subtle differences are hidden within corresponding Definitions 10 (norms) and 12 (arguments).

**Definition 19.** Parabolic addition of vectors is defined by the following formulae:

$$(32) \quad \arg^{(\prime)}(w_1 + w_2) = \frac{\arg^{(\prime)} w_1 \cdot |w_1|^{(\prime)} + \arg^{(\prime)} w_2 \cdot |w_2|^{(\prime)}}{|w_1 + w_2|^{(\prime)}},$$

$$(33) \quad |w_1 + w_2|^{(\prime)} = |w_1|^{(\prime)} \pm |w_2|^{(\prime)},$$

where primed versions are used for the subgroup  $N'$ .

The rule for the norm of sum (33) may look too trivial at the first glance. We should say in its defence that it nicely sits in between the elliptic  $|w + w'| \leq |w| + |w'|$  and hyperbolic  $|w + w'| \geq |w| + |w'|$  triangle inequalities for norms.

The rule (32) for argument of the sum is not arbitrary as well. From the Sine Theorem in the Euclidean geometry we can deduce that:

$$\sin(\phi - \psi') = \frac{|w| \cdot \sin(\psi - \psi')}{|w + w'|}, \quad \sin(\psi' - \phi) = \frac{|w'| \cdot \sin(\psi - \psi')}{|w + w'|},$$

where  $\psi^{(\prime)} = \arg w^{(\prime)}$  and  $\phi = \arg(w + w^{(\prime)})$ . Using parabolic expression (16) for the sine  $\sin \theta = \theta$  we obtain the arguments addition formula (32).

A proper treatment of zeros in denominator of (32) can be achieved through a representation of a dual number  $w = u + \varepsilon v$  as a pair of homogeneous polar coordinates  $[a, r] = [|w|^{(\prime)} \cdot \arg^{(\prime)} w, |w|^{(\prime)}]$  (dashed version for the subgroup  $N'$ ). Then the above addition is defined component-wise in the homogeneous coordinates:

$$w_1 + w_2 = [a_1 + a_2, r_1 + r_2], \quad \text{where } w_i = [a_i, r_i].$$

The multiplication from Defn. 15 is given in the homogeneous polar coordinates by:

$$w_1 \cdot w_2 = [a_1 r_2 + a_2 r_1, r_1 r_2], \quad \text{where } w_i = [a_i, r_i].$$

Thus homogeneous coordinates linearise the addition (32)–(33) and multiplication by a scalar (30). A transition to other more transparent coordinates shall be treated withing birational geometry framework [24].

Both formulae (32)–(33) together uniquely define explicit expressions for addition of vectors. However those expressions are rather cumbersome and not really much needed. Instead we list properties of these operations:

**Proposition 20.** *Vector additions for subgroups  $N$  and  $N'$  defined by (32)–(33) satisfy the following conditions:*

- (1) *They are commutative and associative.*
- (2) *They are distributive for multiplications (28) and (29); consequently:*
- (3) *They are parabolic rotationally invariant;*
- (4) *They are distributive in both ways for the scalar multiplications (30) and (31) respectively:*

$$a \cdot (w_1 + w_2) = a \cdot w_1 + a \cdot w_2, \quad (a + b) \cdot w = a \cdot w + b \cdot w.$$

To complete the construction we need to define the zero vector and the inverse. The inverse of  $w$  has the same argument as  $w$  and the opposite norm.

**Proposition 21.** (N) *The zero vector is  $(0, 0)$  and consequently the inverse of  $(u, v)$  is  $(u, 2u^2 - v)$ .*

(N') *The zero vector is  $(\infty, -1)$  and consequently the inverse of  $(u, v)$  is  $(u, -v - 2)$ .*

Thereafter we can check that scalar multiplications by negative reals are given by the same identities (30) and (31) as for positive ones.

*Remark 22.* The irrelevance of the standard linear structure for parabolic rotations manifests itself in many different ways, e.g. in an apparent “non-conformality” of lengths from parabolic foci, that is with the parameter  $\tilde{\sigma} = 0$  in [15, Prop. 5.12.(iii)]. An adjustment of notions to the proper framework restores the clear picture.

The initial definition of conformality [15, Defn. 5.11] considered the usual limit  $y' \rightarrow y$  along a straight line, i.e. “spoke” in terms of Fig. 1. This is justified in the elliptic and hyperbolic cases. However in the parabolic setting the proper “spokes” are vertical lines, see Fig. 2(P) and (P'), so the limit should be taken along them [15, Prop. 5.13].

## 7. CONCLUSION: INDUCED REPRESENTATIONS

We discussed above various implementations of hypercomplex unitary characters. Now we can return to consideration of induced representations. We can notice that only the subgroup  $K$  requires a complex valued character due to the fact of its compactness. For subgroups  $N'$  and  $A'$  we can consider characters of all three types—elliptic, parabolic and hyperbolic. Moreover a parabolic character can be taken either as algebraic (15) or any of two geometric (22) and (24). Therefore we have seven essentially different induced representations, which multiply types to eleven (counting flavours of parabolic characters).

**Example 23.** Consider the subgroup  $H = K$ , then we are limited to complex valued characters of  $K$  only. All of them are of the form  $\chi_k$ :

$$\chi_k \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = e^{-ikt}, \quad \text{where } k \in \mathbb{Z}.$$

Then the corresponding induced representation (13)  $\rho_k$  act on complex valued functions in the upper half-plane  $\mathbb{R}_+^2 = \text{SL}_2(\mathbb{R})/K$  as follows:

$$(34) \quad \rho_k(g)f(w) = \frac{1}{(cw + d)^k} f\left(\frac{aw + b}{cw + d}\right), \quad \text{where } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

These are unitary representations from the discrete series [26, § IX.2].

**Proposition 24.** *Let  $f_k(w) = (w - i)^k$  for  $k = 2, 3, \dots$ , then*

- (1)  *$f_k$  is an eigenvector for any operator  $\rho_k(h)$ , where  $h \in K$ , with the eigenvalue  $\chi_k(h)$  [26, § IX.2].*
- (2) *The function  $K(z, w) = \rho_k(s(z))f_k(w)$ , where  $s(z)$  is defined in (9), is the Bergman reproducing kernel in the upper half-plane [13, § 3.2].*

Similarly we can get the Cauchy kernel for the limiting case  $k = 1$  of the mock discrete series [26, Ch. IX]. There are many other important connections of representation (34) with complex analysis and operator theory. For example, Möbius transformations of operators lead to Riesz-Dunford functional calculus and associated spectrum [14].

**Example 25.** In the case of the subgroup  $N$  there is a wider choice of possible characters.

- (1) Traditionally only complex valued characters of the subgroup  $N$  are considered, they are:

$$\chi_\tau^{\mathbb{C}} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = e^{i\tau t}, \quad \text{where } \tau \in \mathbb{R}.$$

The corresponding induced representation acts on the space of *complex* valued functions on the upper half-plane  $\mathbb{R}_+^2$ , which is subset of *dual* numbers as a homogeneous space  $\text{SL}_2(\mathbb{R})/N$ . The corresponding formula is

$$\rho_\tau^{\mathbb{C}}(g)f(w) = \exp\left(i\frac{\tau cv}{cu + d}\right) f\left(\frac{aw + b}{cw + d}\right), \quad \text{where } w = u + \varepsilon v, \quad g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The mixture of complex and dual numbers in the same expression is confusing.

- (2) The parabolic character  $\chi_\tau$  with the algebraic flavour is provided by multiplication (15) with the dual number:

$$\chi_\tau \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = e^{\varepsilon \tau t} = 1 + \varepsilon \tau t, \quad \text{where } \tau \in \mathbb{R}.$$

Then the corresponding induced representation is defined on the space of dual numbers valued functions on the upper half-plane of dual numbers by the formula:

$$\rho_\tau(g)f(w) = \left(1 + \varepsilon \frac{\tau cv}{cu + d}\right) f\left(\frac{aw + b}{cw + d}\right),$$

where  $w, \tau$  and  $g$  are as above. This expression contains only dual numbers with their usual algebraic operations. Thus it is linear with respect to them.

- (3) The geometric character  $\chi_\tau^g$  is given by the action (22). Then the corresponding representation acts again on the space of dual numbers valued functions on the upper half-plane of dual numbers as follows:

$$\rho_\tau^g(g)f(w) = \left(1 + \varepsilon \frac{2\tau cv}{cu + d}\right) f\left(\frac{aw + b}{cw + d}\right) + \frac{\tau cv}{cu + d} + \varepsilon \frac{(\tau cv)^2}{(cu + d)^2},$$

where  $w, \tau$  and  $g$  are as above. This representation is linear with respect to operations (30), (32) and (33).

All characters in the previous Example are unitary, the first two in a conventional sense and the last one in the sense of Prop. 18. Then the general scheme of induced representations [12, § 13.2] implies their unitarity in proper senses.

**Theorem 26.** *All three representations of  $SL_2(\mathbb{R})$  from Example 25 are unitary on the space of function on the upper half-plane  $\mathbb{R}_+^2$  of dual numbers with the inner product:*

$$(35) \quad \langle f_1, f_2 \rangle = \int_{\mathbb{R}_+^2} f_1(w) \bar{f}_2(w) \frac{du dv}{v^2}, \quad \text{where } w = u + \varepsilon v,$$

and we use

- (1) the conjugation and multiplication of functions' values in algebras of complex and dual numbers for representations  $\rho_\tau^{\mathbb{C}}$  and  $\rho_\tau$  respectively;
- (2) conjugation (27) and multiplication (28) of functions' values for the representation  $\rho_\tau^{\mathbb{D}}$ .

The inner product (35) is positive defined for the representation  $\rho_\tau^{\mathbb{C}}$  but is not for two others. The respective spaces are parabolic cousins of the Krein spaces [1], which are hyperbolic in our sense.

There are many important questions to be investigated for those induced representations: relations with the three main series of representations (discrete, principal, complementary) of  $SL_2(\mathbb{R})$  [26], connections with various  $\mathfrak{sl}_2$  modules [10, 29], applications to analytic functions [13] and partial differential equations [25], associated functional calculi and spectra [14], etc. These directions can be viewed as parts of the Erlangen programme at large [17, 23] and shall be considered elsewhere.

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## APPENDIX A. OUTPUT OF SYMBOLIC CALCULATIONS

Here are the results of our symbolic calculations. The source code can be obtained from this paper [20] source at <http://arXiv.org>. It uses Clifford algebra facilities [18] of the GiNaC library [2]. The source code is written in noweb [30] literature programming environment.

Calculations for subgroup N and straight spokes

Reference point:  $\begin{pmatrix} 0 & -1 \end{pmatrix}$

Reference point arg: 0

Reference point norm: 1

Cayley of the matrix X:  $\begin{pmatrix} 1 + e^0 x & 1x \\ 0 & 1 - e^0 x \end{pmatrix}$

Rotation by x:  $\begin{pmatrix} u + x & 2ux + v + x^2 \end{pmatrix}$

Rotation of  $(u_0, v_0)$  by  $x$ :  $\begin{pmatrix} x & -1+x^2 \end{pmatrix}$   
 Parabolic norm:  $u^2 - v$   
 Parabolic argument:  $u$   
 Real number  $t_1$  as a dual number:  $\begin{pmatrix} 0 & -t_1 \end{pmatrix}$   
 Product:  $\begin{pmatrix} u' + u & u'^2 + 2u'u + u^2 + v'u^2 - u'^2u^2 - v'v + u'^2v \end{pmatrix}$   
 Product by a scalar:  $\begin{pmatrix} u & u^2 + av - au^2 \end{pmatrix}$   
 Real part:  $\begin{pmatrix} 0 & -uv - u^2 + u^3 + v \end{pmatrix}$   
 Imag part:  $\begin{pmatrix} 1 & 1 + uv - u^3 \end{pmatrix}$   
 Zero vector:  $\begin{pmatrix} 0 & 0 \end{pmatrix}$   
 Negative vector:  $\begin{pmatrix} u & 2u^2 - v \end{pmatrix}$   
 Dual number from argument 0 and norm  $n$ :  $\begin{pmatrix} 0 & -n \end{pmatrix}$   
 Dual number from argument  $a_1$  and norm  $n$ :  $\begin{pmatrix} a_1 & a_1^2 - n \end{pmatrix}$   
 Dual number from argument  $a_1$  and norm  $n$ —norm:  $n$   
 Dual number from argument  $a_1$  and norm  $n$ —arg:  $a_1$   
 Lin comb of two vectors  $a*(1, 0) + b*(-1, 0)$ :  $\begin{pmatrix} -\frac{b-a}{b+a} & \frac{b^2-b^3-a^3-3b^2a+a^2-2ba-3ba^2}{(b+a)^2} \end{pmatrix}$   
 $P$  is the sum  $\Re(P)$  and  $\Im(P)$ : **true**  
 The real part of a real dual number is itself: **true**  
 norm is invariant under rotations: **true**  
 Product is invariant under rotations: **true**  
 Product  $w\bar{w}$  is norm squared: **true**  
 Product  $(u, v) * (u_0, v_0)$  is  $(u, v)$ : **true**  
 Add is commutative: **true**  
 Add is associative: **true**  
 S-mult is commutative: **true**  
 S-mult is associative: **true**  
 S-mult is distributive 1: **true**  
 S-mult is distributive 2: **true**  
 Product is symmetric (commutative): **true**  
 Prod is associative: **true**  
 Product is distributive: **true**

---

Calculations for subgroup  $N'$  and straight spokes

Reference point:  $\begin{pmatrix} \infty & -1 + \infty^2 \end{pmatrix}$   
 Reference point arg: 0  
 Reference point norm: 1  
 Cayley of the matrix  $X$ :  $\begin{pmatrix} 1 - e^0x & 0 \\ 1x & 1 + e^0x \end{pmatrix}$   
 Rotation by  $x$ :  $\begin{pmatrix} \frac{u}{1+ux} & -\frac{2ux+u^2x^2-v}{1+2ux+u^2x^2} \end{pmatrix}$   
 Rotation of  $(u_0, v_0)$  by  $x$ :  $\begin{pmatrix} \frac{1}{x} & -\frac{-1+x^2}{x^2} \end{pmatrix}$   
 Parabolic norm:  $\frac{u^2}{1+v}$   
 Parabolic argument:  $\frac{1}{u}$   
 Real number  $t_1$  as a dual number:  $\begin{pmatrix} \infty & \frac{\infty^2-t_1}{t_1} \end{pmatrix}$   
 Product:  $\begin{pmatrix} \frac{u'u}{u'+u} & \frac{1+v'-u'^2-2u'u-u^2+v'v+v}{(u'+u)^2} \end{pmatrix}$   
 Product by a scalar:  $\begin{pmatrix} u & -\frac{-1+a-v}{a} \end{pmatrix}$   
 Real part:  $\begin{pmatrix} \infty & \frac{\infty^2-u^2+\infty^2v+u}{(-1+u)u} \end{pmatrix}$   
 Imag part:  $\begin{pmatrix} 1 & -\frac{-1+u-v}{u} \end{pmatrix}$   
 Zero vector:  $\begin{pmatrix} \infty & -1 \end{pmatrix}$   
 Negative vector:  $\begin{pmatrix} u & -2 - v \end{pmatrix}$



Dual number from argument 0 and norm n:  $\left( \infty \quad \frac{\infty^2 - n}{n} \right)$

Dual number from argument  $a_1$  and norm n:  $\left( \frac{1}{a_1} \quad -\frac{-1 + a_1^2 n}{a_1^2 n} \right)$

Dual number from argument  $a_1$  and norm n—norm: n

Dual number from argument  $a_1$  and norm n—arg:  $a_1$

Lin comb of two vectors  $a * (1, 0) + b * (-1, 0)$ :  $\left( -\frac{b+a}{b-a} \quad \frac{b-b^2-a^2+2ba+a}{(b-a)^2} \right)$

P is the sum  $\Re(P)$  and  $\Im(P)$ : **true**

The real part of a real dual number is itself: **true**

norm is invariant under rotations: **true**

Product is invariant under rotations: **true**

Product  $w\bar{w}$  is norm squared: **true**

Product  $(u, v) * (u_0, v_0)$  is  $(u, v)$ : **true**

Add is commutative: **true**

Add is associative: **true**

S-mult is commutative: **true**

S-mult is associative: **true**

S-mult is distributive 1: **true**

S-mult is distributive 2: **true**

Product is symmetric (commutative): **true**

Prod is associative: **true**

Product is distributive: **true**

#### Elliptic case of induced representations

map  $r(M)$ :  $\left( \begin{array}{cc} \frac{d^2}{d^2+c^2} & -\frac{cd}{d^2+c^2} \\ \frac{cd}{d^2+c^2} & \frac{d^2}{d^2+c^2} \end{array} \right)$

map  $s^{-1}(M)$ :  $\left( \begin{array}{cc} \frac{1}{d} & \frac{bc^2+c+bd^2}{d^2+c^2} \\ 0 & \frac{d^2}{d^2+c^2} \end{array} \right)$

character:  $\left( \begin{array}{cc} \frac{2cd u + d^2 + c^2 u^2}{2cd u + d^2 + c^2 u^2 + c^2 v^2} & -\frac{c(cu+d)v}{2cd u + d^2 + c^2 u^2 + c^2 v^2} \\ \frac{c(cu+d)v}{2cd u + d^2 + c^2 u^2 + c^2 v^2} & \frac{(cu+d)^2}{2cd u + d^2 + c^2 u^2 + c^2 v^2} \end{array} \right)$

Moebius map:  $\left( \begin{array}{cc} \frac{cav^2+cau^2+bcu+bd+da u}{2cd u + d^2 + c^2 u^2 + c^2 v^2} & -\frac{bcv-dav}{2cd u + d^2 + c^2 u^2 + c^2 v^2} \end{array} \right)$

Moebius map is given by the imaginary unit: **true**

#### Parabolic ( $N'$ ) case of induced representations

map  $r(M)$ :  $\left( \begin{array}{cc} 1 & 0 \\ \frac{c}{d} & 1 \end{array} \right)$

map  $s^{-1}(M)$ :  $\left( \begin{array}{cc} \frac{1}{d} & b \\ 0 & d \end{array} \right)$

character:  $\left( \begin{array}{cc} 1 & 0 \\ \frac{cv}{cu+d} & 1 \end{array} \right)$

Moebius map:  $\left( \begin{array}{cc} \frac{b+au}{cu+d} & -\frac{bcv-dav}{(cu+d)^2} \end{array} \right)$

Moebius map is given by the imaginary unit: **true**

#### Hyperbolic case of induced representations

map  $r(M)$ :  $\left( \begin{array}{cc} \frac{d^2}{d^2-c^2} & \frac{cd}{d^2-c^2} \\ \frac{cd}{d^2-c^2} & \frac{d^2}{d^2-c^2} \end{array} \right)$

map  $s^{-1}(M)$ :  $\left( \begin{array}{cc} \frac{1}{d} & -\frac{bc^2+c-bd^2}{d^2-c^2} \\ 0 & \frac{d^2}{d^2-c^2} \end{array} \right)$



$$\text{character: } \begin{pmatrix} \frac{2cdu+d^2+c^2u^2}{2cdu+d^2+c^2u^2-c^2v^2} & \frac{c(cu+d)v}{2cdu+d^2+c^2u^2-c^2v^2} \\ \frac{c(cu+d)v}{2cdu+d^2+c^2u^2-c^2v^2} & \frac{(cu+d)^2}{2cdu+d^2+c^2u^2-c^2v^2} \end{pmatrix}$$

$$\text{Moebius map: } \begin{pmatrix} -\frac{cav^2-cau^2-bcu-bd-dau}{2cdu+d^2+c^2u^2-c^2v^2} & -\frac{bcv-dav}{2cdu+d^2+c^2u^2-c^2v^2} \end{pmatrix}$$

Moebius map is given by the imaginary unit: **true**

## APPENDIX B. PROGRAM FOR SYMBOLIC CALCULATIONS

This is a documentation for our symbolic calculations supporting this paper. You can obtain the program itself from the [source files](#) of this paper [20] at [arXiv.org](#); L<sup>A</sup>T<sub>E</sub>X compilation of it will produces the file `parab-rotation.nw` in the current directory. This is a `noweb` [30] code of the program. It uses Clifford algebra facilities [18] of the GiNaC library [2].

This piece of software is licensed under [GNU General Public License](#) (Version 3, 29 June 2007) [6].

**B.1. Calculation and Tests.** This Subsection contains code for calculation of various expression. See [16] or GiNaCinfo for usage of Clifford algebra functions.

**B.1.1. Calculation of Expressions.** Firstly, we output the expression of the Cayley transform for a generic element from subgroups  $N$  and  $N'$ .

16a `<Show expressions 16a>+=` (23e) 16b `>`  
`ex XC=canonicalize_clifford((TC*X*TCI).evalm());`  
`formula_out("Cayley of the matrix X: ", XC.subs(sign=0).normal());`

Uses `formula_out` 16c 17h 17h 17h 17h 18a 26d.

Then we calculate Möbius action of those matrix on a point.

16b `<Show expressions 16a>+=` (23e) `<16a 16c>`  
`dual_number W(clifford_moebius_map(XC, P.to_matrix(), e).subs(sign=0).normal()),`  
`W1 = W.subs(1st(u=u1, v=v1));`  
`formula_out("Rotation by x: ", W);`  
`if (not W.is_equal(P.rot(x)))`  
`cout << "*** dualnumber::rot() gives wrong answer *** \\\(" <<`  
`P.rot(x) << "\\)" << endl << endl;`

Uses `formula_out` 16c 17h 17h 17h 17h 18a 26d.

Next we specialise the above result to the reference point.

16c `<Show expressions 16a>+=` (23e) `<16b 16d>`  
`formula_out("Rotation of \\\((u_0, v_0)\\\) by \\\(x\\\)":`  
`W.subs(1st(u = u0, v = v0)).subs(1st(pow(y, -1)).normal().subs(y = 0).normal());`

Defines:

`formula_out`, used in chunks 16, 17, and 21.

The expression for the parabolic norm.

16d `<Show expressions 16a>+=` (23e) `<16c 17a>`  
`formula_out("Parabolic norm: ", P.norm());`  
`formula_out("Parabolic argument: ", P.arg());`

Uses `formula_out` 16c 17h 17h 17h 17h 18a 26d.

Embedding of reals into dual numbers.

17a (Show expressions 16a) +≡ (23e) <16d 17b>  
`possymbol t1("t1","t_1"), a1("a1", "a_1"), n("n");  
formula_out("Real number \\\(t_1\\\) as a dual number: ", dual_number(t1));`

Uses formula\_out 16c 17h 17h 17h 18a 26d.

The expression for the product of two points.

17b (Show expressions 16a) +≡ (23e) <17a 17c>  
`formula_out("Product: ", P*P1);`

Uses formula\_out 16c 17h 17h 17h 18a 26d.

The expression of the product of a point and a scalar.

17c (Show expressions 16a) +≡ (23e) <17b 17d>  
`formula_out("Product by a scalar: ", a*P);`

Uses formula\_out 16c 17h 17h 17h 18a 26d.

Expressions for the real and imaginary parts.

17d (Show expressions 16a) +≡ (23e) <17c 17e>  
`formula_out("Real part: ", P.real_part());  
formula_out("Imag part: ", P.imag_part());`

Uses formula\_out 16c 17h 17h 17h 18a 26d.

The expression for a sum of two points is too cumbersome to be printed.

17e (Show expressions 16a) +≡ (23e) <17c 17f>  
`//formula_out("Add is: ", (P+P1).normal());`

Uses formula\_out 16c 17h 17h 17h 18a 26d.

Expression for zero is

17f (Show expressions 16a) +≡ (23e) <17e 17g>  
`formula_out("Zero vector: ", zero_dual_number());`

Uses formula\_out 16c 17h 17h 17h 18a 26d.

Expression for zero is

17g (Show expressions 16a) +≡ (23e) <17f 17h>  
`formula_out("Negative vector: ", P.neg());`

Uses formula\_out 16c 17h 17h 17h 18a 26d.

Expression for zero is

17h (Show expressions 16a) +≡ (23e) <17g 18a>  
`formula_out("Dual number from argument \\\(0\\\) and norm \\\(n\\\) : ",  
dn_from_arg_mod(0, n));  
dual_number PP=dn_from_arg_mod(a1, n);  
formula_out("Dual number from argument \\\(a_1\\\) and norm \\\(n\\\) : ",  
PP.normal());  
formula_out("Dual number from argument \\\(a_1\\\) and norm \\\(n\\\)---norm: ",  
PP.norm().normal());  
formula_out("Dual number from argument \\\(a_1\\\) and norm \\\(n\\\)---arg: ",  
PP.arg().normal().normal());`

Defines:

formula\_out, used in chunks 16, 17, and 21.

Linear combination of points  $(1, 0)$  and  $(-1, 0)$  with coefficients  $a$  and  $b$ , for the linearisation presented in [21].

18a (Show expressions 16a)  $\vdash \equiv$  (23e)  $\triangleleft$  17h  
`formula_out("Lin comb of two vectors \\\(a*(1, 0)+b*(-1, 0)\\): ",  
 (a*dual_number(1,0)+b*dual_number(-1,0)).normal());`

Defines:

`formula_out`, used in chunks 16, 17, and 21.

B.1.2. *Checking Algebraic Identities.* In this Subsection we verify basic algebraic properties of the defined operations.

A dual number is the sum of its real and imaginary parts.

18b (Check identities 18b)  $\vdash \equiv$  (23e) 18c  $\triangleright$   
`test_out("\\(P\\) is the sum \\\(\\Re(P)\\) and \\\(\\Im(P)\\): ",  
 P-(ex_to<dual_number>(P.real_part())+ex_to<dual_number>(P.imag_part())));`

Defines:

`test_out`, used in chunks 18, 19, and 21f.

A dual number made out of a real  $a$  has the norm of real part equal to  $a$ .

18c (Check identities 18b)  $\vdash \equiv$  (23e)  $\triangleleft$  18b 18d  $\triangleright$   
`test_out("The real part of a real dual number is itself: ",  
 ex_to<dual_number>(dual_number(a).real_part()).norm()-a);`

Defines:

`test_out`, used in chunks 18, 19, and 21f.

The norm is invariant under parabolic rotations, i.e. they are in agreement with Defn. 10.

18d (Check identities 18b)  $\vdash \equiv$  (23e)  $\triangleleft$  18c 18e  $\triangleright$   
`test_out("norm is invariant under rotations: ", P.norm()-W.norm());`

Uses `test_out` 18b 18c 18f 27.

The product  $w_1 \bar{w}_2$  is invariant under rotations, Prop. 3.

18e (Check identities 18b)  $\vdash \equiv$  (23e)  $\triangleleft$  18d 18f  $\triangleright$   
`test_out("Product is invariant under rotations: ", P*P1.conjugate()-W*W1.conjugate());`

Uses `test_out` 18b 18c 18f 27.

Product  $w \bar{w}$  is  $(0, |w|^2)$ , Prop. 4.

18f (Check identities 18b)  $\vdash \equiv$  (23e)  $\triangleleft$  18e 18g  $\triangleright$   
`test_out("Product \\\(w\\bar{w}\\) is norm squared: ",  
 (P*P.conjugate()-dn_from_arg_mod(Arg0, pow(P.norm(), 2))));`

Defines:

`test_out`, used in chunks 18, 19, and 21f.

The reference point is unit under multiplication.

18g (Check identities 18b)  $\vdash \equiv$  (23e)  $\triangleleft$  18f 18h  $\triangleright$   
`test_out("Product \\\((u, v)*(u_0, v_0)\\) is \\\((u, v)\\): ", P*P0-P);`

Uses `test_out` 18b 18c 18f 27.

Addition is commutative, Prop. 1.

18h (Check identities 18b)  $\vdash \equiv$  (23e)  $\triangleleft$  18g 19a  $\triangleright$   
`test_out("Add is commutative: ", (P+P1)-(P1+P));`

Uses `test_out` 18b 18c 18f 27.

Addition is associative, Prop. 1.

19a  $\langle$ Check identities 18b $\rangle + \equiv$  (23e)  $\triangleleft$ 18h 19b $\triangleright$   
`test_out("Add is associative: ", ((P+P1)+ P2)-(P+(P1+P2)));`

Uses test\_out 18b 18c 18f 27.

Multiplication by a scalar is commutative.

19b  $\langle$ Check identities 18b $\rangle + \equiv$  (23e)  $\triangleleft$ 19a 19c $\triangleright$   
`test_out("S-mult is commutative: ", P*a-a*P);`

Uses test\_out 18b 18c 18f 27.

Multiplication by a scalar is associative.

19c  $\langle$ Check identities 18b $\rangle + \equiv$  (23e)  $\triangleleft$ 19b 19d $\triangleright$   
`test_out("S-mult is associative: ", b*P*a-a*P*b);`

Uses test\_out 18b 18c 18f 27.

Distributive law  $a(w_1 + w_2) = aw_1 + aw_2$ , Prop. 4.

19d  $\langle$ Check identities 18b $\rangle + \equiv$  (23e)  $\triangleleft$ 19c 19e $\triangleright$   
`test_out("S-mult is distributive 1: ", a*(P+P1)-(a*P+a*P1));`

Uses test\_out 18b 18c 18f 27.

Distributive law  $(a + b)w = aw + bw$ , Prop. 4.

19e  $\langle$ Check identities 18b $\rangle + \equiv$  (23e)  $\triangleleft$ 19d 19f $\triangleright$   
`test_out("S-mult is distributive 2: ", P*(a+b)-(P*a+P*b));`

Uses test\_out 18b 18c 18f 27.

Product is commutative, Prop. 1.

19f  $\langle$ Check identities 18b $\rangle + \equiv$  (23e)  $\triangleleft$ 19e 19g $\triangleright$   
`test_out("Product is symmetric (commutative): ", P*P1-P1*P);`

Uses test\_out 18b 18c 18f 27.

Product is associative, Prop. 1.

19g  $\langle$ Check identities 18b $\rangle + \equiv$  (23e)  $\triangleleft$ 19f 19h $\triangleright$   
`test_out("Prod is associative: ", (P*P1)*P2-P*(P1*P2));`

Uses test\_out 18b 18c 18f 27.

Product and addition are distributive, Prop. 2.

19h  $\langle$ Check identities 18b $\rangle + \equiv$  (23e)  $\triangleleft$ 19g $\triangleright$   
`test_out("Product is distributive: ", (P+P1)*P2-(P*P2+P1*P2));`

Uses test\_out 18b 18c 18f 27.

**B.2. Induced Representations.** Here we calculate the basic formulae for Section 2.

B.2.1. *Encoded formulae.* This routine encodes the map  $s : \mathbb{R}^2 \rightarrow \text{SL}_2(\mathbb{R})$  (9).

20a (Induced representations routines 20a)  $\equiv$  (22a) 20b  $\triangleright$

```

ex s_map(const ex & u, const ex & v) {
    return matrix(2, 2, lst(v,u,0,1));
}

ex s_map(const ex & P) {
    if (P.nops()  $\equiv$  2)
        return s_map(P.op(0), P.op(1));
    cerr << "s_map() error: parameter should have two operands" << endl;
    return s_map(P,1);
}

```

This routine encodes the map  $r : \text{SL}_2(\mathbb{R}) \rightarrow H$  (12). The first parameter is an element of  $\text{SL}_2(\mathbb{R})$ , the second—is a generic element of subgroup  $H$ . We look specific matrix of the form  $K$  which makes the product  $MK$  belonging to the image of  $s\_map()$ , i.e. its  $(2, 1)$  element should be zero.

20b (Induced representations routines 20a)  $+\equiv$  (22a)  $\triangleleft$  20a 20c  $\triangleright$

```

ex r_map(const ex & M, const ex & K) {
    ex K1=K.evalm(), K2;
    lst vars = (is_a<symbol>(K1.op(2)) ? lst(K1.op(2)) : lst(K1.op(1)));
    if (is_a<symbol>(K1.op(3))) {
        vars = vars.append(K1.op(3));
        K2 = K1.subs(lsolve(lst((M*K1).evalm().op(2) $\equiv$ 0), vars)).subs(K1.op(3) $\equiv$ 1);
    } else
        K2 = K1.subs(lsolve(lst((M*K1).evalm().op(2) $\equiv$ 0), vars));
    return pow(K2, -1).evalm();
}

```

This is the inverse  $s^{-1}$  of the above map  $s$ .

20c (Induced representations routines 20a)  $+\equiv$  (22a)  $\triangleleft$  20b 20d  $\triangleright$

```

ex p_map(const ex & M, const ex & K) {
    ex x = K.evalm().op(3);
    ex MK=(M*pow(r_map(M,K),-1)).evalm();
    ex D=MK.op(3).subs(x $\equiv$ 1).normal();
    return matrix(1, 2, lst((MK.op(1).subs(x $\equiv$ 1).normal() $\div$ D).normal(),
        (MK.op(0).subs(x $\equiv$ 1).normal() $\div$ D).normal()));
}

```

This is a matrix form of the above inverse map  $p\_map()$ .

20d (Induced representations routines 20a)  $+\equiv$  (22a)  $\triangleleft$  20c

```

ex p_map_m(const ex & M, const ex & K) {
    return (M*pow(r_map(M,K),-1)).evalm();
}

```

B.2.2. *Caculation of induced representation formulae.* Firstly we define a generic element  $M$  of  $\text{SL}_2(\mathbb{R})$ .

20e (Induced representations 20e)  $\equiv$  (24d) 20f  $\triangleright$

```

ex M=matrix(2,2, lst(a,b,c,d)), H;

```

We consider the three cases.

20f (Induced representations 20e)  $+\equiv$  (24d)  $\triangleleft$  20e 21a  $\triangleright$

```

string cases[]={"Elliptic", "Parabolic (\\(N^\\prime\\))", "Hyperbolic"};

```

In the those cases *subgroups* holds a generic element of a subgroup  $H$ , see (4), (6) and (7).

21a `<Induced representations 20e>+≡` (24d) `<20f 21b>`  
`ex subgroups=lst(matrix(2, 2, lst(x,-y,y,x)),`  
`matrix(2, 2, lst(1,0,y,1)),`  
`matrix(2, 2, lst(x,y,y,x)));`

Now we run a cycle over the three cases...

21b `<Induced representations 20e>+≡` (24d) `<21a 21c>`  
`for(int i=0; i<3; i++) {`  
`H=subgroups[i];`  
`cout << cases[i] << " case of induced representations\\\\" << endl;`  
`//formula_out("M*H: ", (M*H).evalm());`

Uses formula\_out 16c 17h 17h 17h 17h 18a 26d.

...and output expression of  $r$  (12),...

21c `<Induced representations 20e>+≡` (24d) `<21b 21d>`  
`formula_out("map \\\(r(M)\\): ", r_map(M,H));`

Uses formula\_out 16c 17h 17h 17h 17h 18a 26d.

...matrix form of the inverse  $s^{-1}$  (12),...

21d `<Induced representations 20e>+≡` (24d) `<21c 21e>`  
`formula_out("map \\\(s^{-1}(M)\\): ", p_map_m(M,H).subs(a≡(1+b*c)÷d).normal());`

Uses formula\_out 16c 17h 17h 17h 17h 18a 26d.

...expression for the argument of the character in (13),...

21e `<Induced representations 20e>+≡` (24d) `<21d 21f>`  
`formula_out("character: ", r_map(M*s_map(P),H));`

Uses formula\_out 16c 17h 17h 17h 17h 18a 26d.

...and finally the action (8) of  $SL_2(\mathbb{R})$  on the homogeneous space.

21f `<Induced representations 20e>+≡` (24d) `<21e`  
`formula_out("Moebius map: ", p_map(M*s_map(P.to_matrix()),H));`  
`test_out("Moebius map is given by the imaginary unit: ", p_map(M*s_map(P),H) -`  
`clifford_moebius_map(a*one, b*one, c*one, d*one,P.to_matrix(),e).subs(sign≡i-1));`  
`cout << (latexout ? "\\vspace{2mm}\\hrule" :`  
`"-----" ) << endl;`  
`}`

Uses formula\_out 16c 17h 17h 17h 17h 18a 26d and test\_out 18b 18c 18f 27.

**B.3. Program Outline.** Here is the outline how we use the above parts.

B.3.1. *Test program outline.* Firstly we load **dual\_number** support.

21g `<* 21g>≡` (22a) `>`  
`#include <cycle.h>`  
`#include <dualnum.h>`

The rest of the program makes all checks.

22a  $\langle * 21g \rangle + \equiv$   $\triangleleft 21g$   
 (Definition of variables 22b)  
 (Test routine 23a)  
 (Induced representations routines 20a)  
 (Main procedure 24a)

B.3.2. *Variables.* These **realsymbols** are used in our calculations.

22b (Definition of variables 22b)  $\equiv$  (22a) 22c  $\triangleright$   
**const numeric** half(1,2);

Defines:

**numeric**, used in chunks 25 and 26.

Variables  $v$ s oftenly appear under square roots of the form  $\sqrt{1 + 2v}$ . To facilitate the simplifications of the type  $(\sqrt{1 + 2v})^2 = 1 + 2v$  we (falsely) define them to be positive symbols.

22c (Definition of variables 22b)  $+ \equiv$  (22a)  $\triangleleft 22b \ 22d \triangleright$   
**possymbol** v("v"), v1("v'"), v2("v''");

Other real variables.

22d (Definition of variables 22b)  $+ \equiv$  (22a)  $\triangleleft 22c \ 22e \triangleright$   
**realsymbol** u("u"), u1("u'"), u2("u''"),  
 a("a"), b("b"), c("c"), d("d"), x("x"), y("y"),

Finally this variable keeps the signature of the metric space.

22e (Definition of variables 22b)  $+ \equiv$  (22a)  $\triangleleft 22d \ 22f \triangleright$   
**sign**("s", "\\sigma");

This an index used for the definition of Clifford units.

22f (Definition of variables 22b)  $+ \equiv$  (22a)  $\triangleleft 22e \ 22g \triangleright$   
**varidx** mu(symbol("mu", "\\mu"), 1), nu(symbol("nu", "\\nu"), 2);

Three generic points which are used in calculations.

22g (Definition of variables 22b)  $+ \equiv$  (22a)  $\triangleleft 22f \ 22h \triangleright$   
**dual\_number** P(u, v), P1(u1, v1), P2(u2, v2);

Here we define a parabolic Clifford units *one*, *e0*.

22h (Definition of variables 22b)  $+ \equiv$  (22a)  $\triangleleft 22g$   
**ex** e = clifford\_unit(mu, diag\_matrix(1st(sign))),  
 e0 = e.subs(mu=0),  
 one = dirac.ONE(),  
 e2 = clifford\_unit(nu, diag\_matrix(1st(-1, sign)));



B.3.3. *Test routine.* This routine make the same sequence of checks for both cases of subgroups  $N$  and  $N'$ .

First we define the reference point  $(u0, v0)$ .

23a  $\langle$ Test routine 23a $\rangle \equiv$  (22a) 23b  $\triangleright$

```

void parab_rot_sub(bool N, bool S) {
    cout << "Calculations for subgroup \\ $N$ "
        << (N ? "" : "'") << "\\) and "
        << (S ? "straight" : "geodesics") << " spokes\\\\" << endl;
    set_subgroup(N);
    set_straight_spoke(S);
    ex X,
        W0=dn_from_arg_mod(Arg0, 1),
        u0=W0.op(0),
        v0=W0.op(1),
        P0=matrix(1, 2, lst(u0, v0));

```

Defines:

parab\_rot\_sub, used in chunk 24c.

These two matrices define the Cayley transform and its inverse.

23b  $\langle$ Test routine 23a $\rangle + \equiv$  (22a)  $\triangleleft$  23a 23c  $\triangleright$

```

    cout << "Reference point: \\ $($ " << P0 << "\\)\\\\" << endl;
    cout << "Reference point arg: \\ $($ " << ex_to<dual_number>(W0).arg() << "\\)\\\\" << endl;
    cout << "Reference point norm: \\ $($ " << ex_to<dual_number>(W0).norm() << "\\)\\\\" << endl;

```

For the subgroup  $N$  we consider upper-triangular matrices, for  $N'$ —lower-triangular.

23c  $\langle$ Test routine 23a $\rangle + \equiv$  (22a)  $\triangleleft$  23b 23d  $\triangleright$

```

if (subgroup_N)
    X=matrix(2, 2, lst(one, one*x, 0, one));
else
    X=matrix(2, 2, lst(one, 0, one*x, one));

```

Two different types of Cayley transforms.

23d  $\langle$ Test routine 23a $\rangle + \equiv$  (22a)  $\triangleleft$  23c 23e  $\triangleright$

```

ex TC, TCI;
if (S) {
    TC=matrix(2, 2, lst(one, -e0, -e0, one));
    TCI=matrix(2, 2, lst(one, e0, e0, one));
} else{
    TC=matrix(2, 2, lst(one, -e0*half, -e0*half, one));
    TCI=matrix(2, 2, lst(one, e0*half, e0*half, one));
}

```

Common part of test routine.

23e  $\langle$ Test routine 23a $\rangle + \equiv$  (22a)  $\triangleleft$  23d

```

     $\langle$ Show expressions 16a $\rangle$ 
     $\langle$ Check identities 18b $\rangle$ 
    cout << (latexout ? "\\vspace{2mm}\\hrule" :
        "-----" ) << endl;
}

```

B.3.4. *Main procedure.* It just calls the test routine, calculates the induced representation and draws a few pictures.

We output formulae in  $\text{\LaTeX}$  mode.

24a  $\langle$ Main procedure 24a $\rangle \equiv$  (22a) 24b  $\triangleright$   
`int main(){  
 latexout=true;`

Defines:

`main`, never used.

Preparation of output stream.

24b  $\langle$ Main procedure 24a $\rangle + \equiv$  (22a)  $\triangleleft$  24a 24c  $\triangleright$   
`cout << boolalpha;  
 if (latexout)  
 cout << latex;  
 \langle`Drawing pictures 24f $\rangle$

Now we call the test routine for both  $N$  and  $N'$  subgroups.

24c  $\langle$ Main procedure 24a $\rangle + \equiv$  (22a)  $\triangleleft$  24b 24d  $\triangleright$   
`parab_rot_sub(true, true);  
 parab_rot_sub(false, true);  
 // parab_rot_sub(false, false); To work with geodesic spokes`

Uses `parab_rot_sub` 23a.

Calculation of induced representations formulae.

24d  $\langle$ Main procedure 24a $\rangle + \equiv$  (22a)  $\triangleleft$  24c 24e  $\triangleright$   
 $\langle$ Induced representations 20e $\rangle$

And we finishing by drawing several pictures for Figs. 1 and 2.

24e  $\langle$ Main procedure 24a $\rangle + \equiv$  (22a)  $\triangleleft$  24d  $\triangleright$   
`}`

B.4. **Drawing Orbits.** To draw cycles we use `cycle` library [18].  
 Elliptic orbits (circles).

24f  $\langle$ Drawing pictures 24f $\rangle \equiv$  (24b) 25a  $\triangleright$   
`ofstream asymptote("parab-rot-data.asy");  
 asymptote << "path[] K=";  
 for(int i=0; i<6; i++)  
 cycle2D(lst(0,0),e2.subs(sign=-1),i*i*.04)  
 .asy_path(asymptote, -1.5, 1.5, -2, 2, 0, (i>0));  
 asymptote << ";" << endl;  
 asymptote << "path[] Kb=";  
 cycle2D(lst(0,0),e2.subs(sign=-1),1)  
 .asy_path(asymptote, -1.5, 1.5, -2, 2, 0, false);  
 asymptote << ";" << endl;`

Hyperbolic orbits.

25a

(Drawing pictures 24f) +≡

(24b) ◁ 24f 25b ▷

```
asymptote << "path[] A=";
for(int i=0; i<6; i++) {
  cycle2D(1st(0,0),e2.subs(sign≡1),-i*i*.04)
  .asy_path(asymptote, -1.75, 1.75, -1.5, 2, 0, (i>0));
  cycle2D(1st(0,0),e2.subs(sign≡1),i*i*.12)
  .asy_path(asymptote, -1.75, 1.75, -1.5, 2, 0, true);
}
asymptote << ";" << endl;
asymptote << "path[] Ab=";
cycle2D(1st(0,0),e2.subs(sign≡1),-1)
  .asy_path(asymptote, -1.75, 1.75, -1.5, 2, 0, false);
asymptote << ";" << endl;
```

Hyperbolic orbits for reflected orbits.

25b

(Drawing pictures 24f) +≡

(24b) ◁ 25a 25c ▷

```
asymptote << "path[] At=";
for(int i=0; i<6; i++) {
  cycle2D(1st(0,0),e2.subs(sign≡1),i*i*.04)
  .asy_path(asymptote, -1.75, 1.75, -1.5, 2, 0, (i>0));
  cycle2D(1st(0,0),e2.subs(sign≡1),-i*i*.12)
  .asy_path(asymptote, -1.75, 1.75, -1.5, 2, 0, true);
}
asymptote << ";" << endl;
asymptote << "path[] Abt=";
cycle2D(1st(0,0),e2.subs(sign≡1),1)
  .asy_path(asymptote, -1.75, 1.75, -1.5, 2, 0, false);
asymptote << ";" << endl;
```

Parabolic orbits, subgroup N.

25c

(Drawing pictures 24f) +≡

(24b) ◁ 25b 26a ▷

```
asymptote << "path[] N=";
for(int i=0; i<6; i++)
  cycle2D(1,1st(0,numeric(1,2)),numeric(i,2)-1,e2.subs(sign≡0))
  .asy_path(asymptote, -1.5, 1.5, -2, 2, 0, (i>0));
asymptote << ";" << endl;
asymptote << "path[] Nb=";
cycle2D(1,1st(0,numeric(1,2)),-1,e2.subs(sign≡0))
  .asy_path(asymptote, -1.5, 1.5, -2, 2, 0, false);
asymptote << ";" << endl;
```

Uses numeric 22b.

Parabolic orbits, subgroup  $N'$ .

26a (24b) ◁ 25c 26b ▷

```

(Drawing pictures 24f) +=
  asymptote << "path[] N1=";
  for(int i=0; i<5; i++)
    cycle2D(.5*i*i+1,lst(0,numeric(1,2)),-1,e2.subs(sign=0))
    .asy_path(asymptote, -1.5, 1.5, -1.5, 2, 0, (i>0));
  asymptote << ";" << endl;
  asymptote << "path[] N1b=";
  cycle2D(1,lst(0,numeric(1,2)),-1,e2.subs(sign=0))
  .asy_path(asymptote, -1.5, 1.5, -1.5, 2, 0, false);
  asymptote << ";" << endl;

```

Uses numeric 22b.

Parabolic orbits, subgroup  $N$  geodesic version.

26b (24b) ◁ 26a 26c ▷

```

(Drawing pictures 24f) +=
  asymptote << "path[] Ng=";
  for(int i=0; i<6; i++)
    cycle2D(0.5,lst(0,numeric(1,2)),numeric(i,2)-.5,e2.subs(sign=0))
    .asy_path(asymptote, -1.5, 1.5, -2, 2, 0, (i>0));
  asymptote << ";" << endl;
  asymptote << "path[] Ngb=";
  cycle2D(0.5,lst(0,numeric(1,2)),-.5,e2.subs(sign=0))
  .asy_path(asymptote, -1.5, 1.5, -2, 2, 0, false);
  asymptote << ";" << endl;

```

Uses numeric 22b.

Parabolic orbits, subgroup  $N'$  geodesic version.

26c (24b) ◁ 26b

```

(Drawing pictures 24f) +=
  asymptote << "path[] N1g=";
  for(int i=0; i<5; i++)
    cycle2D(.25*i*i+.5,lst(0,numeric(1,2)),-.5,e2.subs(sign=0))
    .asy_path(asymptote, -1.5, 1.5, -1.5, 2, 0, (i>0));
  asymptote << ";" << endl;
  asymptote << "path[] N1gb=";
  cycle2D(.5,lst(0,numeric(1,2)),-.5,e2.subs(sign=0))
  .asy_path(asymptote, -1.5, 1.5, -1.5, 2, 0, false);
  asymptote << ";" << endl;

  asymptote.close();

```

Uses numeric 22b.

B.4.1. *Output routines.* We use standardised routines to output results of calculations.

26d 27 ▷

```

(Output routines 26d) ≡
void formula_out(string S, const ex & F, bool lineend) {
  cout << S << (latexout ? "\\(" : "(") << F << (latexout ? "\\)" : ")");
  if (lineend)
    cout << (latexout ? "\\\\" : "\\") << endl;
  else
    cout << " ";
}

```

Defines:

formula\_out, used in chunks 16, 17, and 21.

This routine is used to check identities.

27 (Output routines 26d) += < 26d

```

void test_out(string S, const ex & T) {
    cout << S << (latexout ? "\\textbf{" : "*" )
        << (is_a<dual_number>(T) ? ex_to<dual_number>(T).normal().is_zero() :
            T.evalm().normal().is_zero_matrix()) << (latexout ? "}\\\\" : "*" )
        << endl;
}

```

Defines:

test\_out, used in chunks 18, 19, and 21f.

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